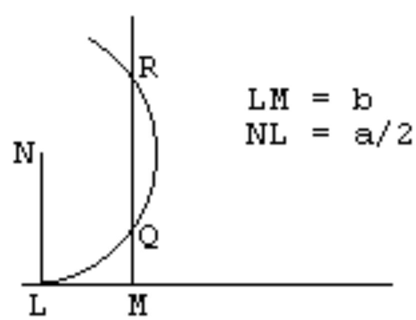
Rhode Island Mathematics Teachers Association

Fall Meeting

October 1, 2009

Mt. St. Charles Academy
Woonsocket, RI



CONSTRUCTIBLE NUMBERS: CONNECTING GEOMETRY AND ALGEBRA

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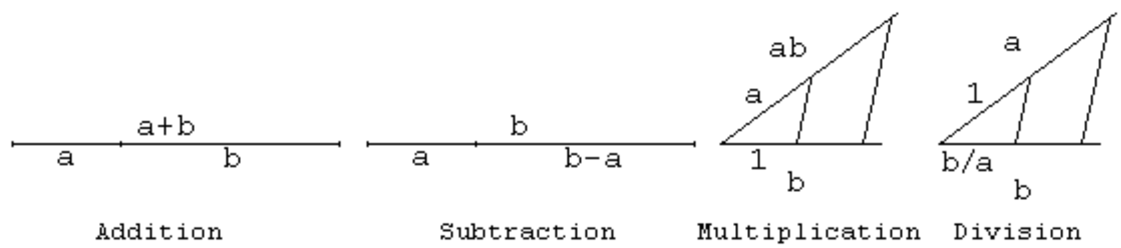
The examples and materials in this workshop are drawn from a variety of sources, such as books on the history of mathematics and several websites, such as “Cut the Knot.” (http://www.cut-the-knot.org/arithmetic/rational.shtml)

Constructible numbers are those whose location on the number line can be constructed with straightedge and compass.

Constructing whole number operations:

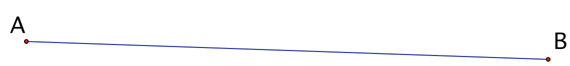


Constructing Fractions:

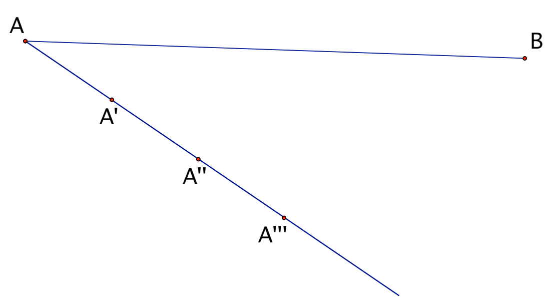
Example: Dividing a length into three equal parts.

1. Draw segment AB
2. Draw a ray from A at any angle
3. Mark off three equal consecutive segments on the ray emanating from point A
4. Draw A'''B
5. Construct lines parallel to A'''B through A'' and A'
6. Construct Points of intersection on segment AB
7. AB has been divided into three equal parts (thirds)

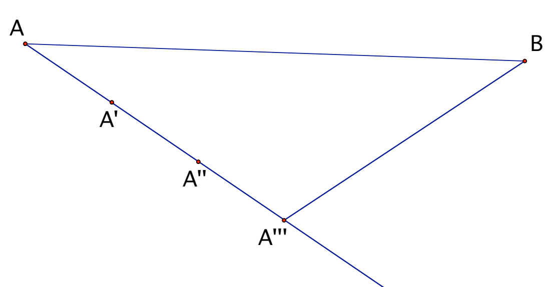
1.



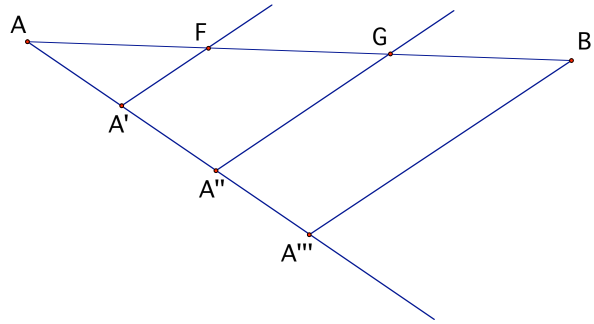
2+3.



4.



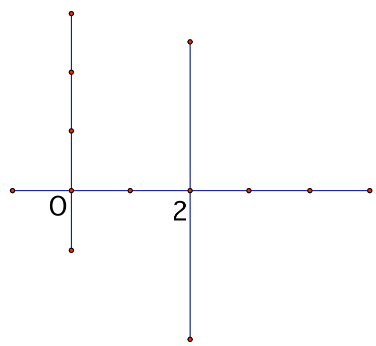
5+6.



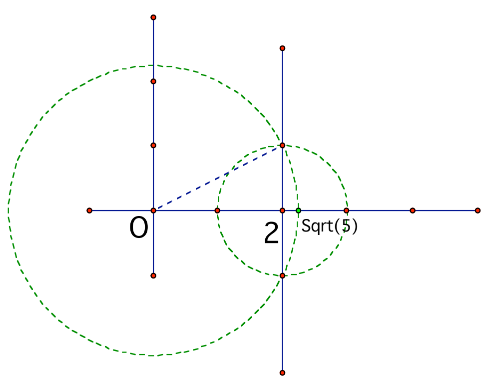
Constructing Irrational Numbers

1. On the number line construct a perpendicular at location 2.
2. Draw a circle with radius 1 and center (2,0).
3. Determine intersection of circle and perpendicular.
4. Draw a circle with center O and radius OB.
5. Determine point of intersection of circle and x-axis. This location is the $\sqrt{5}$ exactly.

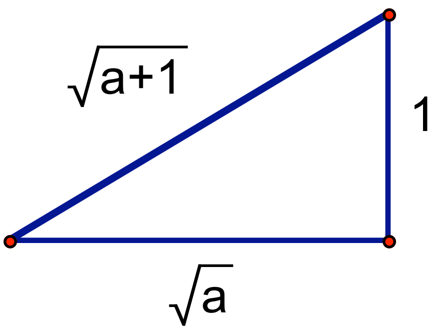
1.

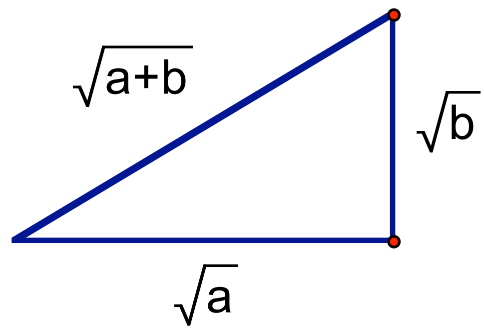


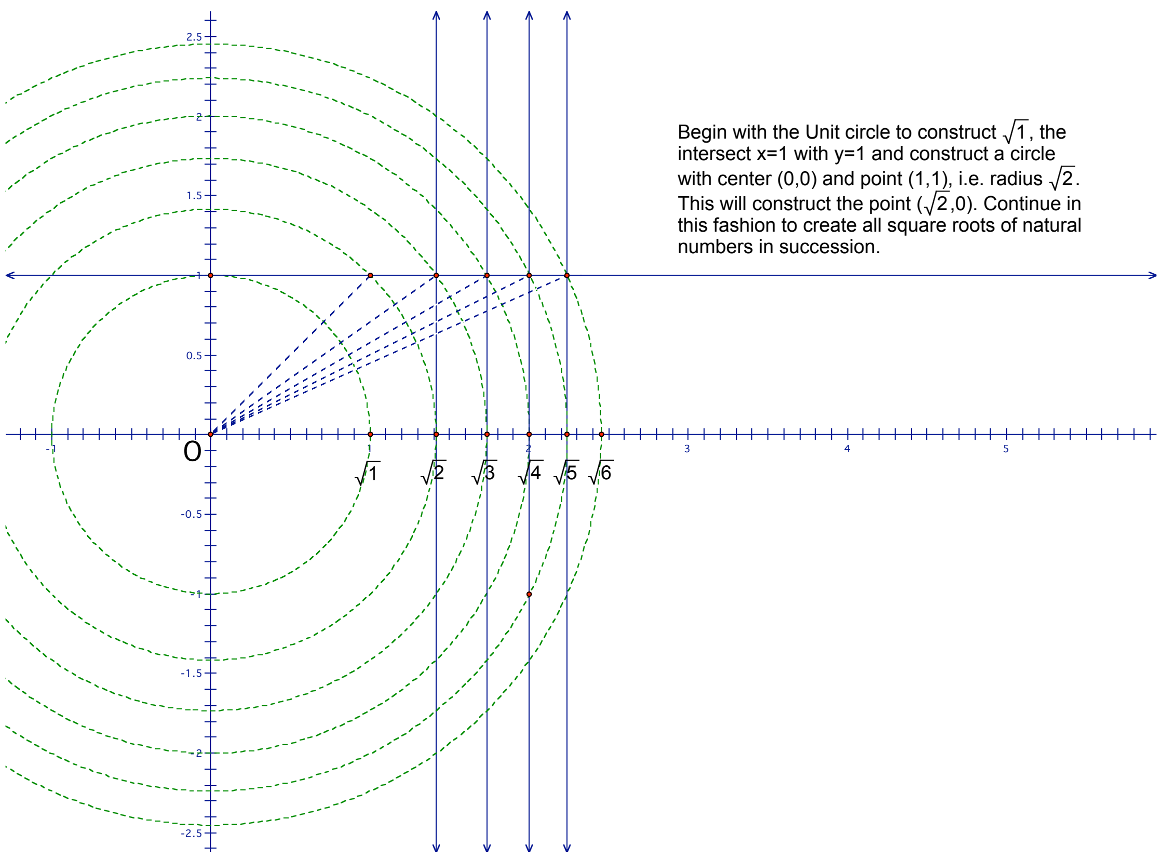
2-5.



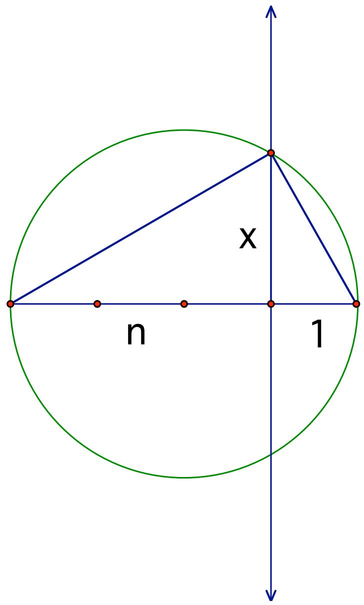
Building a sequence of Square Roots on the number line from 1.







Link with Pythagorean Theorem



In the figure constructed on the left we can use the similarity of two right triangles to calculate the height:

$$\frac{1}{x} = \frac{x}{n} \Rightarrow x^2 = n \Rightarrow x = \sqrt{n}$$

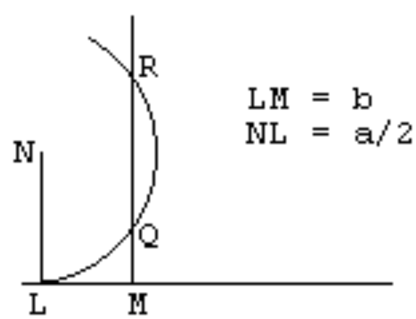
Now that we have a length of $\sqrt{n}$, we can construct other numbers with it, such as: $\sqrt[4]{n}$ or $\sqrt{1+\sqrt{n}}$.

For a detailed discussion, examples, and an interesting applet, see:
http://www.cut-the-knot.org/arithmetic/constructibleExamples.shtml

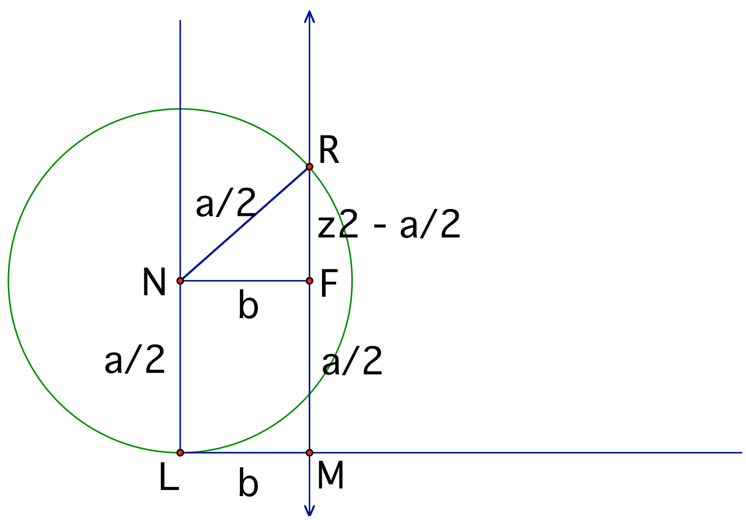
Solving Quadratic Equations ala Descartes (from Cut the Knot)

In the *Geometry*, after pioneering an algebraic notation for an unknown quantity, Descartes gave a geometric solution to the quadratic equation $\mathbf{z^2 = az - b^2}$.
He constructed two perpendicular lines intersecting at point L, and measured NL = a/2 on the vertical line and LM = b on the other.
He then drew a circle with the center at N and radius a/2 and a vertical line at M. When the two lines intersect, as on the diagram, he obtained two solutions to the equation:

$$MR = \frac{a}{2} + \sqrt{\frac{a^2}{4} - b^2} \qquad MQ = \frac{a}{2} - \sqrt{\frac{a^2}{4} - b^2}$$



Proof:



We can apply the Pythagorean theorem in right triangle NFR as follows:

$$\left(\frac{a}{2}\right)^2 = b^2 + \left(z_2 - \frac{a}{2}\right)^2$$

from this follows that

$$z_2^{\ 2} - z_2 a + b^2 = 0$$

and therefore the solution follows from the quadratic formula.